# A Geometric Solution of a Cubic by Omar Khayyam ... in which Coloured Diagrams are Used Instead of Letters for the Greater Ease of Learners

**Deborah A. Kent and David J. Muraki**

**Abstract.** The visual language employed by Oliver Byrne in his 1847 edition of Euclid's *Elements* provides a natural syntax for communicating the geometrical spirit of Omar Khayyam's eleventh-century constructions for solving cubic equations. Inspired by the subtitle (co-opted for this article) from Byrne's *The Elements of Euclid*, we rework one of these constructions by adapting his distinct pictographic style. This graphical presentation removes the modern reliance on algebraic notation and focuses instead on a visualization that emphasizes Khayyam's use of ratios, conic sections, and dimensional reasoning.

**1. INTRODUCTION** For the present-day reader, the world of medieval Islamic algebra is, in many ways, foreign terrain both conceptually and notationally. It is a place where powers of a quantity correspond to actual geometrical dimensions. In other words, $x^2$ literally represents a square area, $x^3$ a cube volume, and what we call a cubic polynomial is expressed as a sum of parallelepiped volumes. To contemporary students, this kind of geometric perspective on polynomial equations may seem strange, and it is tempting to fall back upon the modern framework of algebraic notations and their manipulation. However, these familiar notions make it too easy to read modern ideas into an historical text, thus interfering with the deeper appreciation of a geometric perspective that is a hallmark of medieval Islamic mathematics. To capture more of the eleventh-century geometric spirit of Omar Khayyam's solution to a class of cubic equations, we present here an adaptation using the colorful graphical language of the nineteenth-century educator Oliver Byrne.

Khayyam's proof of his geometric construction involves subtle shifts between lengths, areas, and volumes as he invoked clever ratio arguments on lines, surfaces, and solids. Recasting Khayyam's argument in the pictographs of Byrne removes the obstacle of heavy notation, allows immediate identification of the key geometrical objects, and clarifies their role in the dimensional transitions of the proof. Furthermore, as the Khayyam constructions rely on conic sections, this articulation also highlights the use of their geometrical properties without the algebraic baggage of their Cartesian quadratic equations. Here, we revive Byrne's idiosyncratic vision in a demonstration for how constructive geometry, the arithmetic of ratios and conic sections, work together to produce a geometrical solution to a cubic.

**2. THE VISUAL GEOMETRICAL LANGUAGE OF OLIVER BYRNE** Oliver Byrne (1810-1880) is best remembered for the quirky and magnificent volume *The First Six Books of The Elements of Euclid in which Coloured Diagrams and Symbols are Used Instead of Letters for the Greater Ease of Learners* that first appeared in 1847 [3]. The work has recently enjoyed a resurgence due to the facsimile published by Taschen in 2010 and again in 2013 [12], [13]. Byrne's novel idea was to present Euclidean geometry graphically, using diagrams printed in brilliant primary colours instead of the more conventional — and, Byrne thought, cumbersome — system of

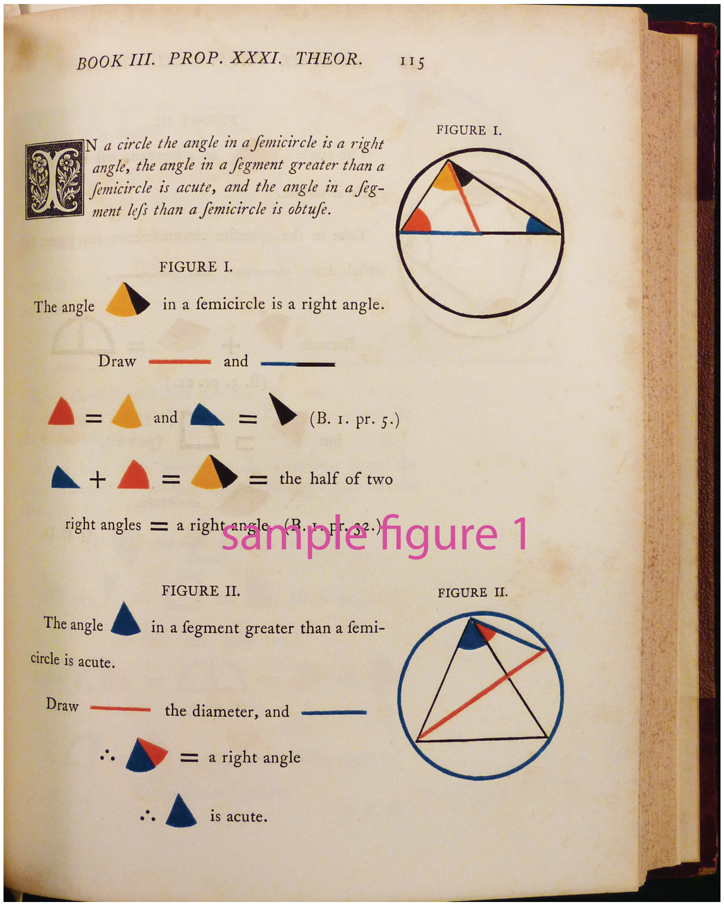


**Figure 1.** A page showing Book III, Proposition 31 from Oliver Byrne's colorful version of Euclid's *Elements*. In *Victorian Book Design and Colour Printing*, Ruari McLean describes Byrne's pages as, "a unique riot of red, yellow and blue; on some pages letters and numbers only are printed in colour, sprinkled over the page like tiny wild flowers, demanding the most meticulous register: elsewhere, solid squares, triangles and circles are printed in gaudy and theatrical colors, attaining a verve not seen again on book pages till the days of Dufy, Matisse, and Derain" [14, p. 70]. Used with permission from University of British Columbia Special Collections.

heavily labelled figures and densely-worded proofs still used in many Geometry texts today.

Little is known of Byrne's life or education. The publication record depicts him as a prolific author of over one hundred books that indicate his pedagogical and mathematical interests were closely tied to engineering and surveying with a particular focus on facilitating cognitive and computational efficiency for students in areas of calculation, mensuration, and geometry. The frontispieces of these books label Byrne as a Professor of Mathematics at the Putney College for Civil Engineers and a consulting actuary to the Philanthropic Life Assurance Society, as well as a civil, military, and mechanical engineer. Byrne worked at the privately funded (and somewhat fringe institution) Putney College from its 1840 opening until financial trouble forced closure in 1857 [16, p. 273]. After his tenure there, Byrne served as a surveyor of British settlements in the Falkland Islands. He also continued to write books, some of which champion him as inventor of patented calculating instruments and of "The system of facilitating the acquirement of geometry, and other lineal arts and sciences by coloured diagrams"

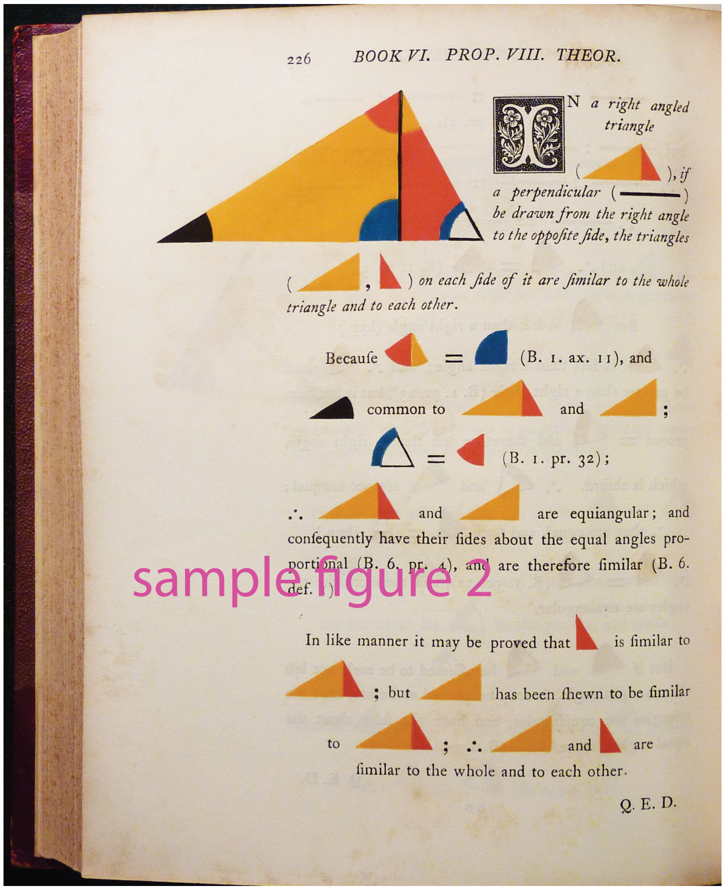


**Figure 2.** A page showing Book VI, Proposition 8 from Oliver Byrne's colourful version of Euclid's *Elements*. Whittingham's spectacular craftsmanship is evident in the registration of the four, hand-printed colours. Byrne's radical pedagogical technique mixes with antique Victorian initials and medial s's that were already 50 years out of date in Byrne's day. The minimalist "layout and primary-color palette red, blue, yellow, black . . . prefigures the art and design of 20th-century avant-garde movements" [9, p. B12]. Used with permission from University of British Columbia Special Collections.

[2, cover page].

Byrne likely developed this pedagogical system out of his frustration with standard methods of geometrical instruction. His first published effort to make Euclid's *Elements* more accessible to students appeared in 1841, titled *The Doctrine of Proportion clearly developed, on a comprehensive, original, and very easy system; or, the Fifth book of Euclid simplified* [2]. The preface reads like a disgruntled instructor reciting a litany of shortcomings of existing nineteenth-century texts on the subject of proportions. Byrne's goal, then, was to articulate proportion algebraically, arithmetically, *and* geometrically, and to "endeavour to clear, without destroying the universality and rigor of its conclusions, this extensive mathematical branch of that difficult, elaborate, and intricate reasoning with which the prevailing opinion has so long charged it" [2, p. xviii - xix].

Byrne initially envisioned a book printed with differently coloured pictographs, but colour printing was prohibitively expensive. He instead advised readers to highlight the figures with coloured pencils (Byrne, 1841, xvi). Still dedicated to his pedagogical vision, Byrne later teamed up with innovative publisher William Pickering and printer

Charles Whittingham, an influential duo in nineteenth-century British book production. This collaboration brought Byrne's colourful vision to life in the 1847 volume [3]. In the text, Byrne relies on colors and shapes to communicate propositions and proofs of Euclid's *Elements* in pictographs to facilitate learning. Byrne claimed that this approach enabled students to learn the *Elements* of Euclid "in less than one third the time usually employed, and retention of this memory is much more permanent" [3, p. ix].

Despite these ambitions, Byrne's volume does not appear to have revolutionized geometrical instruction. David Eugene Smith mentions Byrne's *Euclid* in a footnote to his 1915 edition of Augustus De Morgan's *A Budget of Paradoxes* [4], where he writes that "[t]here is some merit in speaking of the red triangle instead of the triangle $ABC$, but not enough to give the method any standing" [4, p. 329]. Smith's remark matches De Morgan's reported dismissal of Byrne's book as a novelty. Dozens of unsold copies were auctioned off at Pickering's bankruptcy sale just a few years after the publication. The bankruptcy is attributed to default on a loan Pickering had guaranteed [14, p. 13], but the great production expense and slow sales of Byrne's volume probably didn't help the situation. The fortunes of Byrne's *Euclid* changed in the twentieth century. It was featured as a remarkable volume in Ruari McLean's seminal work *Victorian Book Design and Colour Printing* [14] and is now valued at five-figure prices by modern collectors. The contents of Byrne's book, too, found favour in Edward Tufte's work *The Visual Display of Quantitative Information* [17]. Byrne's streamlined visual presentation style seems appropriate to current sensibilities, digital illustration software, and online publication. In the following, we adapt his pictographic scheme for purposes of facilitating in modern readers a greater appreciation for the geometric thinking of eleventh-century mathematicians.

**3. THE GEOMETRICAL SOLUTIONS OF OMAR KHAYYAM** Abū'l-Fath Ghiyāth al-Dīn 'Umar ibn Ibrāhīm al-Khayyāmī al-Nīshāpūrī, usually known in English as Omar Khayyam, is best remembered as an astronomer and as the poet of the *Rubā'iyāt*, but he also published mathematical and philsophical works in a region near present-day Afghanistan. His treatise on algebra, *Risāla fī al-jabr wa al-muqābala*, is one surviving mathematical work. A first translation into English by Daoud Kasir [10] was published in 1931, with the most recent in 2000 by Roshdi Rashed and Bijan Vahabzadeh [15] under the auspices of UNESCO. This major work by Khayyam is his comprehensive study of constructing solutions for what modern readers know as polynomials from linear through cubic degrees.

At the beginning of the *Algebra*, Khayyam explains how the lineage of topics in his book starts with Archimedes. By the end of the ninth century, equations involving squares and cubes were known to Islamic mathematicians who had access to newly translated Greek mathematics, including work by Euclid, Archimedes, and Apollonius. They also would have known about the three classic Greek construction problems. Islamic mathematicians are generally credited with advances in algebra, developing and advancing Hindu and Babylonian work [11, p. 271]. This work was heavily influenced by the Islamic mathematicians' exposure to Greek geometric texts and, more specifically, the idea that a mathematical problem was not fully solved without a proof. And proofs, it seemed, were geometric. Part of the project of medieval Islamic mathematics was to justify algebraic rules through geometry [11, p. 271]. One of the earliest Islamic algebra texts, by ninth-century mathematician al-Khwārizmī, includes geometric proofs for algebraic algorithms for solving quadratic equations. In the tenth and eleventh centuries, Islamic mathematicians subsequently solved some of Archimedes' cubic equations using intersecting conic sections [11, p. 287], [15, p. 111]. Khayyam

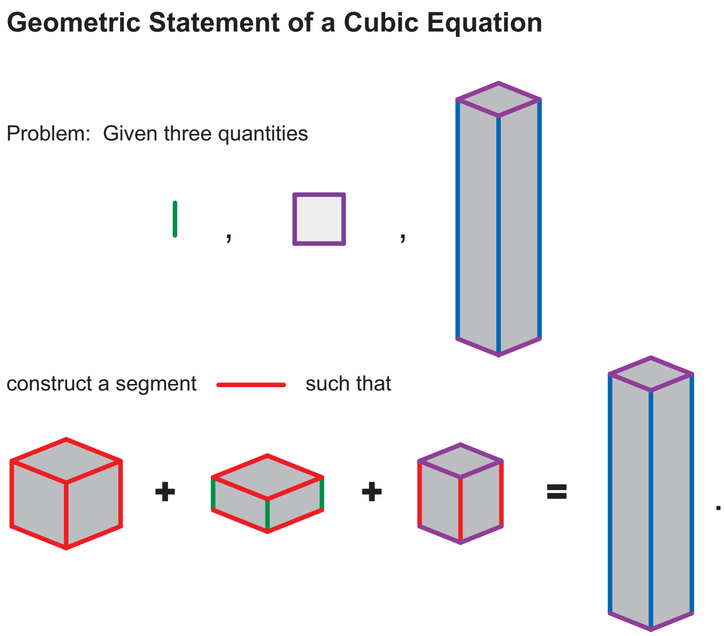


**Figure 3.** Khayyam's cubic problem "a cube and squares and sides equal to a number" presented in the style of Oliver Byrne. Given the green, purple and blue segments, the red segment is the unknown that is to be constructed.

would thus have been familiar with classical Greek mathematics as well as work from his Islamic predecessors. Particularly, Khayyam clearly states in his preface that Euclid's *Elements*, the *Data*, and the first two chapters of Apollonius' *Conics* are necessary prerequisites, and to the best of his ability, sufficient references for understanding the contents of the *Algebra*.

For Khayyam, numbers could only be positive, which meant there were fourteen different types of cubics. For each cubic, he introduced the conic sections necessary for the solution and proved that the constructed solution was indeed correct. Khayyam viewed algebra as a method "to determine the numerical and the geometrical unknowns" [15, p. 111]. From the outset, he aimed to supplement numerical solutions by geometric construction — which he did for both first and second degree polynomials. However, Khayyam concluded that numerical solutions for cubic equations are "impossible," but using the intersection of conic sections he provided geometric solutions for all cases.

This paper illustrates one case of Khayyam's construction. It is the same as that featured in *The History of Mathematics: A Reader* [11] as an excerpt from the *Algebra*. The specific construction results in a segment that gives the solution to the cubic problem shown in Figure 3 and described as "a cube and squares and sides are equal to a number" [15, p. 141]. Note that "cube" corresponds to a modern $x^3$ term, and "squares" and "sides" would be, for us, multiples of $x^2$ and $x$. A modern formulation would then introduce coefficients to give a cubic polynomial, but in Khayyam's time, each term in this equation instead corresponded to a three-dimensional volume. Notably, a quartic, or any higher-order polynomial, would not exist in this reality [15, p. 114].

For Khayyam, the unknown quantity ($x$) represented a line segment that could be used to build three boxes so that their combined volumes equalled a given value. The first box is a perfect cube whose edges are the unknown ($x^3$). The second box has square base with the unknown as edges, but a given height ($ax^2$). The third box has unknown height and a given square base ($b^2x$). A solution, then, would be a segment such that the total volume of these three boxes must be equal to a given volume ($b^2c$), which is described as having the same square base as the third box. The contemporary notation for this Khayyam case would be the cubic equation $x^3 + ax^2 + b^2x = b^2c$ with positive coefficients. There is only one positive solution for this particular case.

Constructing a line segment of a length to accomplish this constitutes a geometric solution to the cubic equation. It is a bit of a challenge to follow the full narrative of Khayyam's construction and proof in prose — introducing endpoint labels and using line-segment notation is only a slight improvement. We begin our graphical retelling by recasting the cubic problem as in Figure 3, which serves as an introduction to our Byrne-like pictographs.

The three given quantities are each represented by a geometrical element shown in Figure 3. Line segments with like color and line-style are congruent, so this equation with four volumes is precisely the cubic problem described in the preceding paragraph. The green line segment is the given height of the second box, and in modern language, the coefficient on the square term. Likewise, the area of the square with purple edges is the coefficient on the linear term, and the dark shaded solid has a volume that is the polynomial constant. By our color convention, the base of the given volume has the same area as the purple-edged square. The geometric solution of this cubic would be a construction that produces the red line segment which satisfies the volume equation.

For the figures that follow, additional graphical conventions are introduced. Black is used for neutral segments and curves with no implication of congruence. All two-dimensional area quantities are identified by light gray shading, with no implication of equality. All three-dimensional objects are rendered in perspective and their volumes are identified by dark gray shading.

## 4. TWO EQUAL-AREA RESULTS

Khayyam's geometric solution for "a cube and squares and sides are equal to a number" relies on a pair of equal area results, one from Euclid's *Elements* [6] and one from Apollonius' *Conics* [8]. Figure 4 presents these lemmas in pictographic form. The first equal area lemma is Euclid's *Elements*, Book VI, Proposition 13.[1] In a semi-circle with given diameter, the square of a perpendicular altitude is equal in area to a rectangle whose edges are pieces of the diameter delineated by the altitude. A geometrical proof of this result is shown in the next section not for completeness, but as an introductory demonstration of a proof with our Byrne-style interpretation.

The second equal area result comes from Apollonius' *Conics*, Book II, Proposition 12 [8].[2] Apollonius actually proves a more general result for the constancy of a product of distances from a hyperbola to its asymptotes, however, only the special case for the rectangular hyperbola is needed here. Although we will not prove it, this case is familiar to students who recognize $xy = c$ as the Cartesian equation of a rectangular hyperbola. In Figure 4, the red and blue dots indicate points on the hyperbola, whose asymptotes are the black dotted lines. The statement that the areas are equal implies that the products of their edge lengths must be equal. The choice of colours used in Figure 4 correspond to those that appear later in Khayyam's proof (Figures 7 and 8).

Figure 5 presents a proof of the semi-circle lemma as inspired by Oliver Byrne's pictographic style. It differs from Byrne's proof of Book VI, Proposition 13 [3, p. 231] as it replaces his notion of mean proportional with more familiar language of similar

[1] Note that this is a version of Euclid Book II, Proposition 14, which is equivalent to the classical extraction of a square root [6, p. 216].

[2] This is more readily available in [7, p. 59].

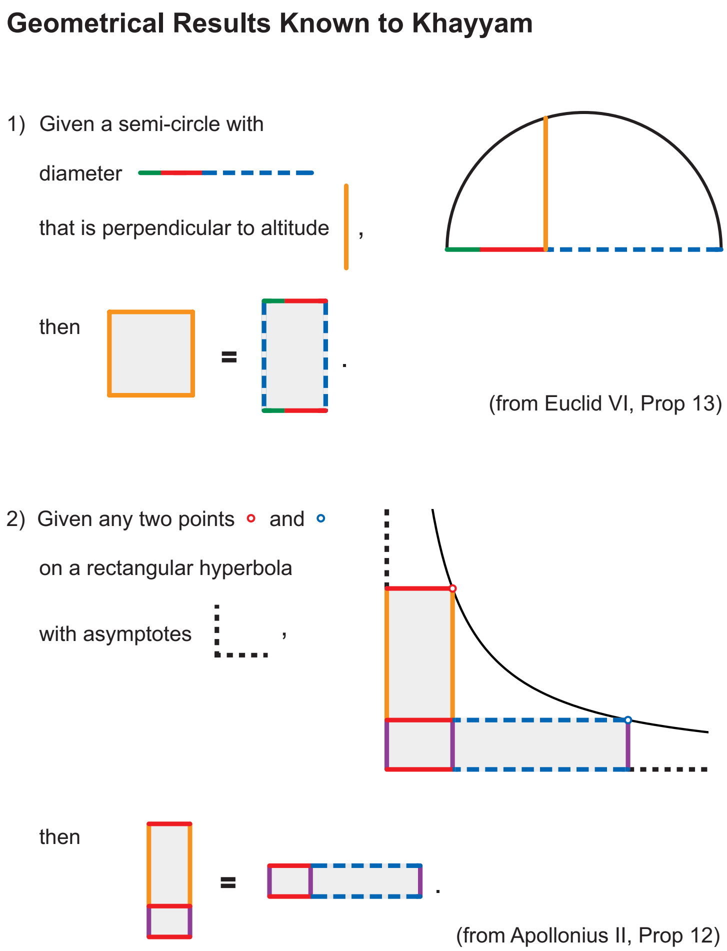


**Figure 4.** Two prerequisite lemmas for Khayyam's proof presented in the style of Oliver Byrne. The first is the classical extraction of a square root presented in Euclid's *Elements* [6]. The second is a fact about hyperbolas from Apollonius' *Conics* [8]. The colour choices here foreshadow those appearing in Figures 7 and 8.

triangles.

Starting with the construction in Figure 4, connect the top of the altitude to each of the endpoints of the diameter. Proposition 31 from Book III of Euclid's *Elements* (Figure 1) ensures the triangle formed is a right triangle. Then Proposition 8 from Book VI (Figure 2) shows that all three triangles are similar triangles. This proof uses the similarity of the two interior triangles that share the orange altitude. As this altitude joins the diameter at a right angle, the ratio of the green+red segment to the orange altitude is equal to the ratio of the orange altitude to the blue dashed segment. The fourth step in Figure 5 is essentially geometric cross multiplication since the product of two line segments is the area of a rectangle. As the above ratios are equal, then the area of the square with orange altitude edge length is equal to the area of the rectangle whose edges are the blue dotted segment and the green+red segment.

With these two lemmas in hand, and some facility interpreting the Byrne-style figures, we proceed to Khayyam's constructive solution of a cubic.

**5. KHAYYAM'S CONSTRUCTION** In Khayyam's presentation, he merges directions to construct a line segment that is a solution to a polynomial of the form "a cube

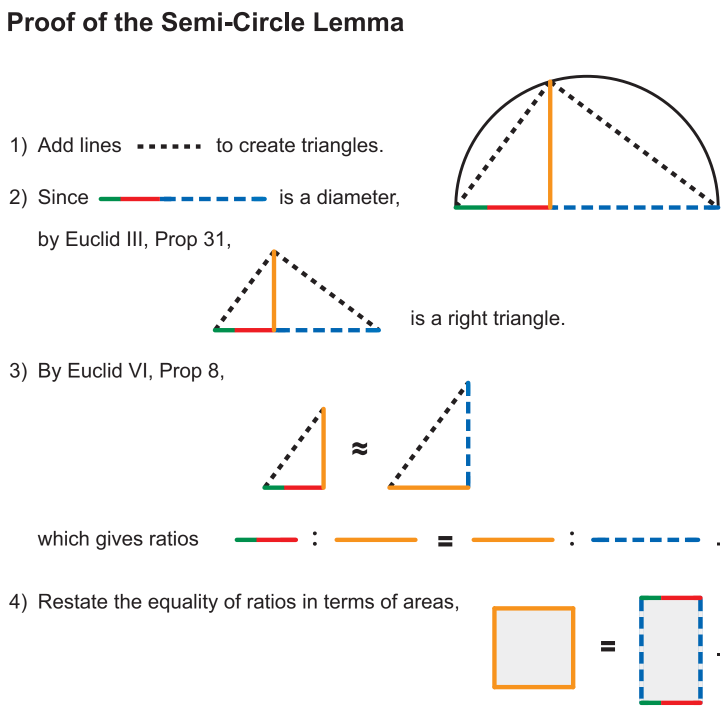


**Figure 5.** An illustrative proof of the semi-circle lemma from Figure 4 presented in the style of Oliver Byrne. The fourth step is an example of shifting between the dimensions of quantities that is common in Khayyam's arguments. Specifically, the ratio in 3) means that the products of the edges are equal, so the rectangles in 4) have the same area.

and squares and sides are equal to a number" together with a justification that the constructed segment is a solution [15, p. 141-142]. For clarity here, the construction has been extracted and presented first in Figure 6, then Figures 7 and 8 prove how the constructed segment satisfies the stated cubic. It is unimportant to the exposition, but readers may be interested to know that the diagrams in Figures 3-8 are precisely scaled using the cubic $x^3 + \frac{8}{16}x^2 + \frac{9}{16}x = \frac{33}{16}$ with $x = 1$.

Recall from Figure 3 that one line segment, one area, and one volume are given in the problem. (In modern algebraic terminology, these would be the given coefficients.) From these quantities, the green, purple, and blue line segments are known. Assemble these line segments as shown in Figure 6. Then construct a semi-circle with green+blue as a diameter. The purple line segment locates asymptotes for a rectangular hyperbola through the blue endpoint of the diameter. Book II, Proposition 4 from Apollonius' *Conics* states that a hyperbola is determined by asymptotes and a point [8, p. 156-157].[3] The red point is the second intersection of the hyperbola and the circle, and the horizontal segment connecting it with the vertical asymptote is the constructed solution to the cubic.

**6. KHAYYAM'S PROOF** Our proof in Figures 7 and 8 begins by introducing to the construction of Figure 6 two rectangles which immediately lead to the use of the hyperbola lemma of Figure 4. The pictographs are left to tell the rest of the story, but it is noteworthy that four changes of dimension occur in Khayyam's proof.[4]

[3] Also in [7, p. 56].
[4] Interested learners could reflect on the case "cube + sides = squares + a number."

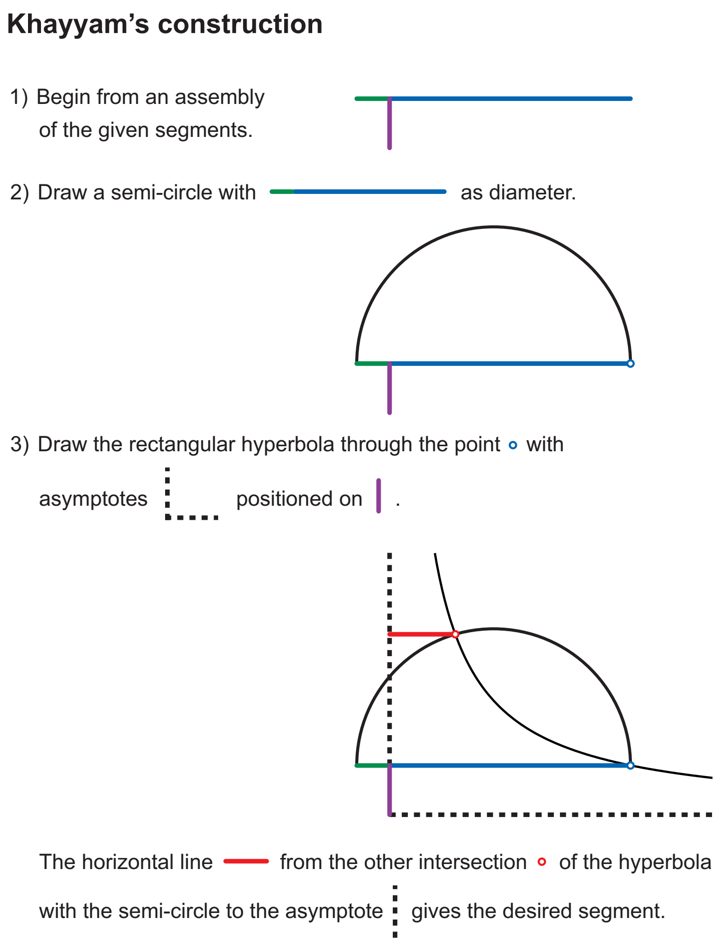


**Figure 6.** Khayyam's construction of a solution to a cubic. The green, purple and blue segments are the givens as shown in Figure 3. The intersection point produced by the semi-circle and the hyperbola determine the desired red segment.

**7. CONCLUSION** A geometric solution to a cubic equation may seem peculiar to modern eyes, but the study of cubic equations (and indeed much of medieval algebra) was initially motivated by geometric problems. Modern readers tend to lack fluency in reading proofs of this type without rewriting them in familiar algebraic notation. While that can offer valuable insight, such translation sometimes obscures certain features of historical mathematics in context. Adopting a presentation style similar to that of Oliver Byrne — one that minimizes labeling and dense prose to rely instead on colour and space — showcases the geometric nature of Khayyam's construction of cubic solutions. This approach also highlights the relationship of earlier geometric work — on conic sections, on ratios, on doubling the cube — with the projects of medieval Islamic algebra. In a mathematical culture where powers of $x$ literally corresponded to geometrical dimensions, solving cubic equations marked a significant achievement. Although Khayyam presented constructions for geometric solutions to all types of cubic equations, he was nevertheless explicitly aware that the arithmetic problem of these cubics was still unsolved. This task remained open until solved by Gerolamo Cardano in the mid-sixteenth century.

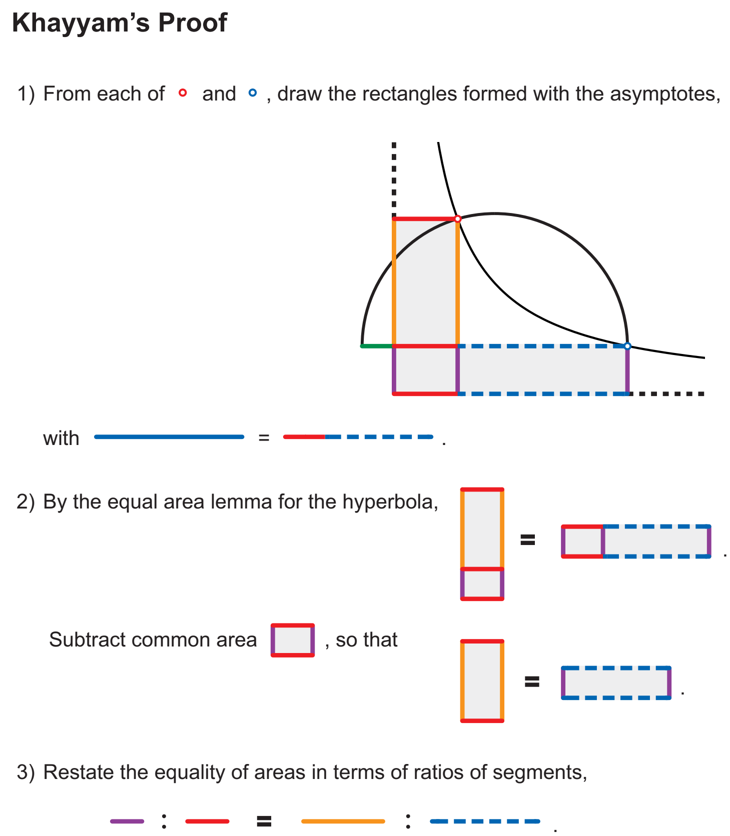


**Figure 7.** Khayyam's proof in the style of Oliver Byrne (part 1 of 2).

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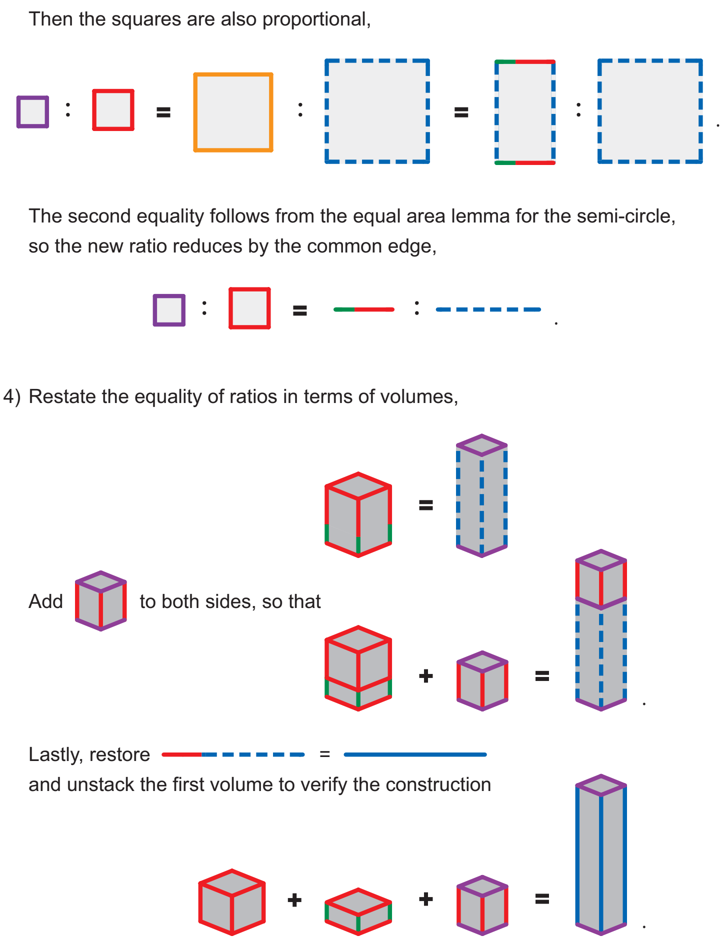


**Figure 8.** Khayyam's proof in the style of Oliver Byrne (part 2 of 2).

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